Birational Geometry Seminar

17th March 2023

Regularizations of positive entropy automorphisms

Let X, Y be smooth projective varieties $/\mathbb{C}$.

Def: (1) $f : X \dashrightarrow Y$ is a birational map, if
$\exists\, U \subset X,\ V \subset Y$ - Zariski open and $f|_U : U \xrightarrow{\simeq} V$

(2) If $f : X \dashrightarrow Y$ is birational, then the graph of f is

$$\Gamma_f = \overline{\{(x,y) \mid x \in U,\ y = f|_U(x)\}} \subset X \times Y \quad X \xleftarrow{p_1} \cdot \xrightarrow{p_2} Y$$

(3) $\mathrm{Ind}(f) = \{x \in X \mid \dim(p_1^{-1}(x)) > 0\}$ - indeterminacy locus

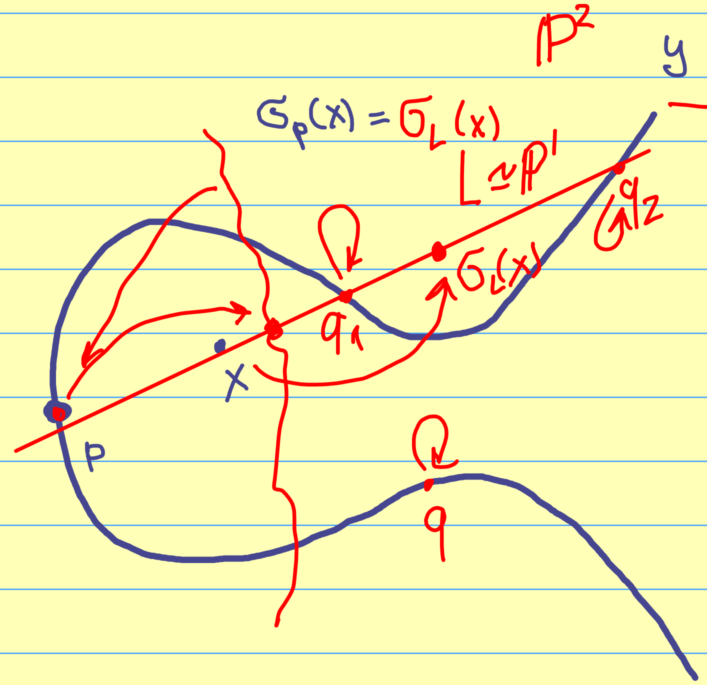
(4) $\mathrm{Exc}(f) = \{x \in X \mid p_2|_{\Gamma_f}$ is not an iso in $p_1^{-1}(x)\}$ - exceptional locus

(5) $\mathrm{Bir}(X) = \{f : X \dashrightarrow X \mid f$ - birational map$\}$
- group of birational automorphisms of X.

Example of a birational automorphism of $\mathbb{P}^n$

Fix $Y \subset \mathbb{P}^n$ – smooth cubic hypersurface.

$p \in Y \rightsquigarrow \sigma_p : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$



Where goes general point $x \in \mathbb{P}^n$?

(1) Fix line $\overset{\simeq \mathbb{P}^1}{L} = \langle p, x \rangle \subset \mathbb{P}^n$

(2) $L \cap Y = \{p, q_1, q_2\}$

(3) There exists a non-trivial
$\sigma_L : L \longrightarrow L$
s.t. $\sigma_L(q_i) = q_i$ and $\sigma_L^2 = Id_L$

Fact: if q is a general point on Y then $\sigma_p(q) = q$.

Let $(x_0 : x_1 : \ldots : x_n)$ be coordinates on $\mathbb{P}^n$ and

$\deg P_i = i$

$$Y = \{x_0^2 P_1(x_1 \ldots x_n) + x_0 P_2(x_1 \ldots x_n) + P_3(x_1 \ldots x_n) = 0\}$$
$$\cup\!\!\!\shortmid$$
$$p = (1:0:\ldots:0)$$

$$\sigma_p(x_0 : x_1 : \ldots : x_n) = (x_0 P_2 + 2P_3 : -x_1(P_2 + 2x_0 P_1) : \ldots : -x_n(P_2 + 2x_0 P_1))$$

$$\mathrm{Ind}(\sigma_p) = \{q \in Y \mid \text{line } \langle q, p\rangle \text{ is tangent to } Y \text{ in } q\} \overset{\ni p}{=}$$

$$= \{x_0 P_2 + 2P_3 = P_2 + 2x_0 P_1 = 0\}$$

Question: X - smooth projective variety.
Given $f: X \dashrightarrow X$ birational automorphism.
Is it regularizable ?

I.e. is there Y - smooth projective variety
and $\alpha: X \dashrightarrow Y$ - birational map
s.t. $\alpha \circ f \circ \alpha^{-1} \in \mathrm{Aut}(Y)$?

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed, "\alpha"'] & X \arrow[d, dashed, "\alpha"] \\ Y \arrow[r, "\alpha f \alpha^{-1}"] & Y \end{tikzcd}$$

Examples: cases when $f: X \dashrightarrow X$ is regularizable.

- $f^n = \mathrm{Id}_X \Rightarrow f$ is regularizable. $G \subset \mathrm{Bir}(X)$
  In fact, if f^n is regularizable $\Rightarrow f$ is regularizable.

- Weil: $f \in G \subset \mathrm{Bir}(X)$, where G is connected Lie group
  $\Rightarrow f$ is regularizable.

- $\dim(X) = 1 \Rightarrow \mathrm{Bir}(X) = \mathrm{Aut}(X)$

- $\dim(X) = 2 \Rightarrow \varkappa(X) \geq 0 \Rightarrow X$ admits a minimal model X_{min}
  $\mathrm{Bir}(X) = \mathrm{Aut}(X_{min})$
  $\varkappa(X) = -\infty \Rightarrow$ there are non-regularizable aut.

There are criterions which relate the dynamical properties of f with the property of being regularizable.

- $\dim(X) = 3 \Rightarrow$?..

Dynamical properties of birational automorphisms

codim Z + codim W = dim X
$Z, W \rightsquigarrow (Z, W)$. intersection

X - smooth projective variety $\Rightarrow$ $N^i(X) = \{\sum_{j=1}^m a_j Z_j \mid \operatorname{codim} Z_j = i\} /_{\sim_{num}}$, $a_j \in \mathbb{R}$

$i=1$: $N^1(X) = NS(X)$

$f: X \dashrightarrow X \quad \rightsquigarrow \quad f^*: N^i(X) \longrightarrow N^i(X)$

$$f^* Z = \delta_* p^* Z \in \bigoplus_i N^i(X) \qquad \begin{tikzcd} & V \arrow[dl, "\delta"'] \arrow[dr, "p"] & \\ X \arrow[rr, dashed, "f"] & & X \end{tikzcd}$$

Def: If $H \in N^1(X)$ is an ample class s.t. $H^{\dim X} = 1$ then define i-th degree of f:

$H \cdot C = (H, C) > 0$, C - curve

$$\deg_{i,H}(f) = \underbrace{f^*(H^i)}_{\in N^i(X)} \cdot \underbrace{H^{\dim(X)-i}}_{\in N^{\dim X - i}} \in N^{\dim X}(X) \simeq \mathbb{R}$$

Thm (Dinh, Sibony)

There exists a limit $\lambda_i(f) = \lim_{n\to\infty} \left(\deg_{i,H}(f^n)\right)^{\frac{1}{n}}$.

Numbers $\lambda_i(f)$ does not depend on H and bir model of (X, f).

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, dashed] & X \arrow[d, dashed] \\ X' \arrow[r, dashed, "f'"] & X' \end{tikzcd}$$

Def: $\lambda_i(f)$ is i-th dynamical degree of $f: X \dashrightarrow X$.

Properties:

- $\lambda_0(f) = \lambda_{\dim(X)}(f) = 1$. ($f^*[X]$, $f^*\,pt$)
- Log - concavity: $\lambda_i(f)^2 \geq \lambda_{i-1}(f) \cdot \lambda_{i+1}(f)$, $1 \leq i \leq \dim X - 1$
- Gromov - Yomdin: Let f be regular aut of X.
  Then $h_{top}(f) = \log\left(\max_{1 \leq i \leq \dim X}\{\lambda_i(f)\}\right)$,

  where $h_{top}(f)$ is the topological entropy of f.

Criterions for regularizability for surface automorphisms

X - smooth projective surface, $f: X \dashrightarrow X$ bir. automorphism.
$\lambda_0(f) = \lambda_2(f) = 1$, $\lambda_1(f) \geq 1$.

Discuss the case $\lambda_1(f) > 1$.

Thm (Blanc, Cantat, 2016) $f: X \dashrightarrow X$
If $\lambda_1(f) > 1$ then there are three options for it:

- $\lambda_1(f)$ is a Salem number.
  (i.e. all its Galois conj are $\underbrace{\frac{1}{\lambda_1(f)}}$, $\alpha_1, \dots, \alpha_m$ where $m \geq 1$, $|\alpha_i| = 1$)
  f is regularizable in this case.
- $\lambda_1(f)$ is a Pisot non-quadratic number.
  (i.e. all its Galois conj are $\underbrace{\alpha_1 \dots \alpha_m}$, where $m \geq 2$, $|\alpha_i| < 1$
  f is non-regularizable in this case.
- $\lambda_1(f)$ is a quadratic number.

ExE $\circlearrowleft$ $SL(\mathbb{Z})$ There examples of regularizable and non-regularizable f with such dynamical degree

Thm (Diller, Favre, 2001) $f: X \dashrightarrow X$ $(\lambda_1(f) > 1)$

(1) There exists a birational model $\tilde{f}: \tilde{X} \dashrightarrow \tilde{X}$ such that
$$(\tilde{f}^n)^* = (\tilde{f}^*)^n : N^1(\tilde{X}) \to N^1(\tilde{X})$$

(2) $\lambda_1(f)$ is a simple eigenvalue of $f^*: N^1(\tilde{X}) \to N^1(\tilde{X})$

(3) f is regularizable iff $\Theta^2 = 0$, where Θ is the eigenvector of $f^*: N^1(X) \to N^1(X)$ with eigenvalue $\lambda_1(f)$.

Example: of (positive entropy) surface automorphism.
$\lambda_1(f) > 1$

Fix $Y \subset \mathbb{P}^2$ smooth cubic curve and fix $p_1, \dots, p_k \in Y$

$p_i \rightsquigarrow \sigma_{p_i} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational involution.

$$f = \sigma_{p_1} \circ \sigma_{p_2} \circ \dots \circ \sigma_{p_k} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

Thm (Blanc 2008)
$f = \sigma_{p_1} \circ \sigma_{p_2} \circ \dots \circ \sigma_{p_k}$ is a regularizable automorphism.

If $p_1, \dots, p_k$ are sufficiently general and if $k \geq 3$ then
$\lambda_1(f) > 1$

WHAT HAPPENS IN HIGHER DIMENSIONS ?

Let $Y \subset \mathbb{P}^3$ be a smooth cubic surface; $p_1, p_2, p_3 \in Y$

$$f = \sigma_{p_1} \circ \sigma_{p_2} \circ \sigma_{p_3} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

Lemma (Blanc) There is a pseudo-automorphism model of f and if p_1, p_2, p_3 are general then $\lambda_1(f) > 1$.

$f^*\theta = \lambda_1(f)\theta \rightsquigarrow \theta \in N^1(X)$

Thm (K.) $\subset \mathbb{P}^3 / \mathbb{C}$
Let Y be a very general cubic surface and $p_1, p_2, p_3 \in Y$ general.
Then $\sigma_{p_1} \circ \sigma_{p_2} \circ \sigma_{p_3}$ is non-regularizable.

$\mathbb{P}^n \dashrightarrow \mathbb{P}^n \quad (x_0 \dots x_n) \mapsto \left(\frac{1}{x_0} \dots \frac{1}{x_n}\right)$

Geometry of $\sigma_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$

Let $Y \subset \mathbb{P}^3$ be a smooth cubic surface, $p \in Y$

$$\sigma_p : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$$

$\mathrm{Ind}(\sigma_p) = \{ q \in Y \mid \text{line } \langle p, q \rangle \text{ is tangent to } Y \text{ in } q \}$

$$\ni p \in \Gamma_p \text{ - curve in } Y$$

Lemma: Let $X = \mathrm{Bl}_{p,\Gamma_p} \mathbb{P}^3$. Then σ_p regularizes on X.

$$\begin{tikzcd}
X \arrow[r, "\sigma_{p,X}"] \arrow[d, "\mathrm{Bl}_{p,\Gamma_p}"'] & X \arrow[d, "\mathrm{Bl}_{p,\Gamma_p}"] \\
\mathbb{P}^3 \arrow[r, dashed, "\sigma_p"'] & \mathbb{P}^3
\end{tikzcd}$$

Denote by $\tilde{Y}$ the proper preimage of Y in X. Then

$$\sigma_{p,X}|_{\tilde{Y}} = \mathrm{Id}_{\tilde{Y}} : \tilde{Y} \to \tilde{Y}$$

Now let $p_1, p_2 \in Y$ and let $\Gamma_i = \mathrm{Ind}(\sigma_{p_i})$ for $i = 1, 2$.

Γ_1, Γ_2 - curves on the surface $Y \Rightarrow \Gamma_1 \cap \Gamma_2 \neq \emptyset$.

$$\begin{tikzcd}
\mathrm{Bl}_{p_1,p_2,\Gamma_1,\Gamma_2}\mathbb{P}^3 \arrow[r, "\sigma_1"] \arrow[d] & \mathrm{Bl}_{p_1,p_2,\Gamma_1,\Gamma_2}\mathbb{P}^3 \arrow[r, dashed, "\text{flop}"] \arrow[d] & \mathrm{Bl}_{p_1,p_2,\Gamma_2,\Gamma_1}\mathbb{P}^3 \arrow[r, "\sigma_2"] \arrow[d] & \mathrm{Bl}_{p_1,p_2,\Gamma_2,\Gamma_1}\mathbb{P}^3 \arrow[d] \\
\mathrm{Bl}_{p_1,p_2,\Gamma_1}\mathbb{P}^3 \arrow[r, "\sigma_1"] \arrow[d] & \mathrm{Bl}_{p_1,p_2,\Gamma_1}\mathbb{P}^3 \arrow[r, dashed, "\text{Atiyah}"] \arrow[d] & \mathrm{Bl}_{p_1,p_2,\Gamma_2}\mathbb{P}^3 \arrow[r, "\sigma_2"] \arrow[d] & \mathrm{Bl}_{p_1,p_2,\Gamma_2}\mathbb{P}^3 \arrow[d] \\
\mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_1}"'] & \mathbb{P}^3 \arrow[r, equal] & \mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_2}"'] & \mathbb{P}^3
\end{tikzcd}$$

Criterion for a pseudo-automorphism

Let $Y \subset \mathbb{P}^3$ be a surface; $p_1, p_2, p_3 \in Y$

$$X = Bl_{p_1,p_2,p_3,\Gamma_1\Gamma_2\Gamma_3} \mathbb{P}^3 \xrightarrow[\delta]{} \mathbb{P}^3$$

$$\tilde{\sigma}_i = \delta^{-1} \circ \sigma_{p_i} \circ \delta : X \dashrightarrow X$$

Lemma: $f = \tilde{\sigma}_1 \circ \tilde{\sigma}_2 \circ \tilde{\sigma}_3 : X \dashrightarrow X.$

Both automorphisms f and f^{-1} don't contract divisors. $\overset{\text{def}}{\Longleftrightarrow}$ f is pseudo-automorphism

$$(f^n)^* = (f^*)^n : N^1(X) \longrightarrow N^1(X).$$

Thm (Truong)

Assume that $f : X \dashrightarrow X$ is a pseudo-automorphism and $\lambda_1(f)^2 > \lambda_2(f)$.

Then

(1) $\lambda_1(f)$ is a simple eigenvalue of $f^* : N^1(X) \to N^1(X)$

(2) Let H be an ample class on X. Then

$$N^1(X) \ni \Theta = \lim_{n\to\infty} ((f^n)^* H)/\lambda_1(f)^n \neq 0$$

is a pseudo-effective class such that $f^* \Theta = \lambda_1(f)\, \Theta$

Thm (K)

Let X be a smooth projective threefold and let $f : X \dashrightarrow X$ be a pseudo-automorphism which satisfies the following condition:

(1) $\lambda_1(f)^2 > \lambda_2(f).$ $\quad \lambda_1(f) > 1 \Rightarrow \exists\, \Theta \quad f^*\Theta = \lambda_1 \cdot \Theta$ $\quad (\lambda_1(f)^2 \geq \lambda_2(f) \cdot \lambda_0(f)$ by log-concavit

(2) There exists a curve C such that $C \cdot \Theta < 0$.

(3) $C \not\subset \mathrm{Ind}(f^{-m})$ for infinitely many $m > 0$.

Then f is non-regularizable.

Composition of three involutions.

Let $Y \subset \mathbb{P}^3$ be a smooth cubic surface; $p_1, p_2, p_3 \in Y$

Let $X_i = Bl_{p_1, p_2, p_3, \Gamma_i, \Gamma_{i+1}, \Gamma_{i+2}} \mathbb{P}^3 \xrightarrow[\delta_i]{} \mathbb{P}^3$

$$\begin{tikzcd}
X_1 \arrow[r, "\tilde{\sigma}_1"] \arrow[d, "\delta_1"'] \arrow[rrrrrrr, dashed, bend left, "F"] & X_1 \arrow[r, dashed, "flop"] \arrow[d, "\delta_1"] & X_2 \arrow[r, "\tilde{\sigma}_2"] \arrow[d, "\delta_2"'] & X_2 \arrow[r, dashed, "flop"] \arrow[d, "\delta_2"] & X_3 \arrow[r, "\tilde{\sigma}_3"] \arrow[d, "\delta_3"'] & X_3 \arrow[r, dashed, "flop"] \arrow[d, "\delta_3"] & X_1 \arrow[d, "\delta_1"] \\
\mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_1}"'] & \mathbb{P}^3 \arrow[r, equal] & \mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_2}"'] & \mathbb{P}^3 \arrow[r, equal] & \mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_3}"'] & \mathbb{P}^3 \arrow[r, equal] & \mathbb{P}^3
\end{tikzcd}$$

Our goal is to show that F is non-regularizable

We have to check the following conditions for F:

→ (1) $\lambda_1(F)^2 > \lambda_2(F)$.

→ (2) There exists a curve C such that $C \cdot \Theta < 0$.

→ (3) $C \not\subset \operatorname{Ind}(F^{-m})$ for infinitely many $m > 0$.

(1) $\lambda_1(F)^2 > \lambda_2(F)$ follows from Blanc's computation of dynamical degrees for F.

(2) There exists a curve C such that $C \cdot \Theta < 0$.

$$\begin{tikzcd}
C \arrow[r, phantom, "\subset"] \arrow[d] & X_1 \arrow[r, "\tilde{\sigma}_1"] \arrow[d, "\delta_1"'] & X_1 \arrow[r, dashed, "flop"'] \arrow[d, "\delta_1"] & X_2 \arrow[d, "\delta_2"] \\
L \arrow[r, phantom, "\subset"] & \mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_1}"] & \mathbb{P}^3 \arrow[r, equal] & \mathbb{P}^3
\end{tikzcd}$$

$\tilde{\sigma}_1(C) \subset \operatorname{Ind}(flop)$

Let $q \in \mathbb{P}^3$ be a point in $\Gamma_1 \cap \Gamma_2$

(then lines $\langle p_1, q\rangle$ and $\langle p_2, q\rangle$ are tangent to Y in q)

Let $L = \langle p_1, q\rangle$ is a line in $\mathbb{P}^3$

Let C be a proper preimage of L to $\mathbb{P}^3$.

The last condition

$$\begin{tikzcd} C \arrow[d] & \subset & X_1 \arrow[r, "\sigma_1", "\sim"'] \arrow[d, "\delta_1"'] & X_1 \arrow[r, dashed] \arrow[d, "\delta_1"] & X_2 \arrow[d, "\delta_2"'] & \\ L & \subset & \mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_1}"] & \mathbb{P}^3 \arrow[r, equal] & \mathbb{P}^3 \arrow[r, dashed, "\sigma_{p_3}\circ\sigma_{p_2}"'] & \mathbb{P}^3 \end{tikzcd}$$

(3) $C \not\subset \mathrm{Ind}(F^{-m})$ for ALL $m>0$.

We prove an equivalent condition that $L \not\subset \mathrm{Ind}(f^{-m})$ for $m>0$.

Step 1: Choose coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbb{P}^3$ such that

$p_1 = (0:1:0:0)$ $\quad p_2 = (0:0:1:0)$ $\quad p_3 = (0:0:0:1)$ $\quad L = \{x_2 = x_3 = 0\}$

$$Y = \left\{ \sum_{|I|=3} a_I x^I = 0 \right\} \quad \text{where } a_I \in \mathbb{C}$$

Then σ_{p_i} are given by formulas depending on a_I.

Step 2: Fix $P = (X : Y : 0 : 0) \in L$ and compute $f^{-m}(P) = P_m$

$$P_m = (R_0(X,Y,a_I) : R_1(X,Y,a_I) : R_2(X,Y,a_I) : R_3(X,Y,a_I))$$

where R_i are polynomials of X, Y and a_I.

Computing in SAGE modulo an ideal J

$$J = (8,\ 4(a_{2001}a_{0210} + a_{1101}a_{1110}),\ a_I)_{I\in A} \subset \mathbb{Z}[X, Y, a_I]$$

Lemma: Polynomials R_0, R_1, R_2, R_3 are never zero elements in $\mathbb{Z}[X, Y, a_I]$.

Step 3: if a_I, X, Y are very general in $\mathbb{C}$, then $(X:Y:0:0) \notin \mathrm{Ind}(f^{-m})$

$\Rightarrow L \not\subset \mathrm{Ind}(f^{-m}) \quad \forall m>0$.

Thank you!